Generalized Lifshitz-Kosevich scaling at quantum criticality from the holographic correspondence

Sean A. Hartnoll* and Diego M. Hofman†
Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
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We characterize quantum oscillations in the magnetic susceptibility of a quantum critical non-Fermi liquid. The computation is performed in a strongly interacting regime using the nonperturbative holographic correspondence. The temperature dependence of the amplitude of the oscillations is shown to depend on a critical exponent ν. For general ν the temperature scaling is distinct from the textbook Lifshitz-Kosevich formula. At the "marginal" value $\nu=\frac{1}{2}$, the Lifshitz-Kosevich formula is recovered despite strong interactions. As a by-product of our analysis we present a formalism for computing the amplitude of quantum oscillations for general fermionic theories very efficiently.

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I. RESULTS AND BACKGROUND

A central question in the theoretical characterization of non-Fermi liquids is the fate of the Fermi surface. For instance the "strange metal," and perhaps quantum critical, regions of the cuprate or heavy fermion phase diagrams separate phases with very distinct energy-momentum distributions of fermions. This is seen in many experimental probes, a recent discussion with strong overlap with the concerns of the present paper is Ref. 1.

Quantum oscillations are a robust feature of systems with a Fermi surface.[2] The recent and ongoing experimental observation of quantum oscillations in the copper oxide high-temperature superconductors[3–9] is reinvigorating theoretical approaches to the subject (e.g., Refs. 10 and 11). Present measurements are, perhaps surprisingly, consistent with textbook results for quantum oscillations in Fermi liquids. However, as an increasing range of regimes are investigated, in these and other quintessentially non-Fermi liquid materials, it will be crucial to have theoretical templates available for comparison. For instance, the exciting recent results of Ref. 12 show that the effective quasiparticle mass, as read off by fitting quantum oscillations to the Fermi liquid formula, appears to diverge as one approaches a metal-insulator quantum phase transition. A similar divergence is observed in heavy fermion compounds.[13]

It has long been suspected that strong electronic correlations should lead to deviations from the established Fermi liquid results for quantum oscillations; recent work investigating the effect of interactions on quantum oscillations includes.[14–18] The theoretical hurdle we attempt to address is that the most interesting regimes are often strongly coupled and perturbative quantum field theory treatments may not fully capture the physics of interest. In this paper we will use the inherently nonperturbative "holographic correspondence" (see e.g., Refs. 19 and 20 for relevant introductions) to give a controlled computation of quantum oscillations in the magnetic susceptibility of a strongly interacting quantum critical non-Fermi liquid. We will however highlight similarities with the approach in Ref. 14.

The main result of this paper will be the following expression for the leading period de Haas-van Alphen magnetic oscillations in a class of 2+1 dimensional theories that exhibit an emergent quantum criticality at low energies

$$\chi_{\text{osc.}} = -\frac{\partial^2 \Omega_{\text{osc.}}}{\partial B^2} = \frac{\pi A T c k_F^4}{eB^3}\cos\frac{\pi c k_F^2}{eB}\sum_{n=0}^{\infty} e^{-cT/eBk_F^2/\mu(T/\mu)^{2\nu-1}F_n(\nu)}, \tag{1}$$

where χ is the magnetic susceptibility, e is the charge of a fermionic operator, A is the area of the sample, T the temperature, c the speed of light, k_F the Fermi momentum, B the applied magnetic field, μ the chemical potential, and ν a critical exponent. Our computations are in the clean limit, with no disorder. The most important of these parameters, for our purposes, is the critical exponent ν, which satisfies $0 \le \nu \le \frac{1}{2}$. At $\nu=\frac{1}{2}$ we will find $F_n(\frac{1}{2})=2\pi^2\bar{h}(n+\frac{1}{2})$, where $\bar{h}$ is a dimensionless constant defined below. The sum in expression (1) then gives

$$\chi_{\text{osc.}} = \frac{\pi A T c k_F^4}{2eB^3}\frac{\cos\frac{\pi c k_F^2}{eB}}{\sinh\frac{\pi^2 \bar{h} c T}{eB}\frac{k_F^2}{\mu}}. \quad \left(\nu=\frac{1}{2}\right). \tag{2}$$

This is essentially the textbook Lifshitz-Kosevich result,[2,21] as we discuss in more detail below. Our theories will be in 2+1 dimensions, although many results can likely be generalized to 3+1 dimensions. When $\nu<\frac{1}{2}$ the functions $F_n(\nu)$, given below, are considerably more complicated. The point we wish to emphasize, however, is that at larger temperatures $T \gtrsim \frac{\mu eB}{ck_F^2}$, the decay of the amplitude as a function of T is not of the simple exponential form predicted by the Lifshitz-Kosevich formula, but rather

$$\chi_{\text{osc.}} \sim e^{-T^{2\nu}}. \tag{3}$$

This is what we will mean by a generalized Lifshitz-Kosevich scaling. If we write this scaling as a temperature-dependent effective quasiparticle mass in the usual Lifshitz-Kosevich formula, then

$$m_\star \sim \frac{k_F^2}{\mu}\left(\frac{\mu}{T}\right)^{1-2\nu}, \tag{4}$$

which is divergent when $T \ll \mu$ and $\nu<\frac{1}{2}$. This is perhaps interesting in the light of the observations in Refs. 12 and 13.

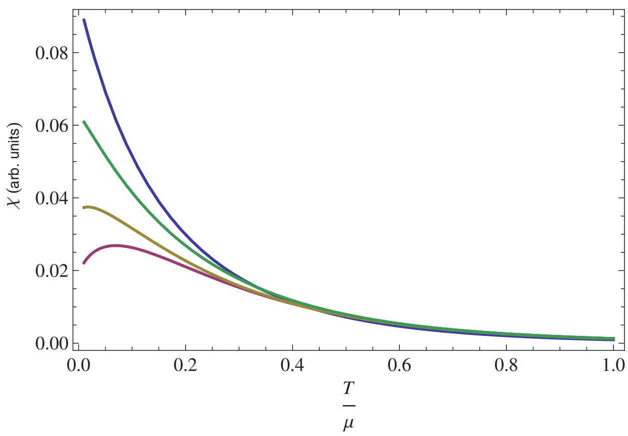

FIG. 1. (Color online) Typical dependences of the amplitude of quantum oscillations on temperature. For illustration $\nu=\frac{1}{3}$, $\frac{eB}{ck_F^2}=1$, and $\alpha=1$. Angles of $\hat{h}$ from top to bottom: $\varphi=\{-\varphi_0, -0.2\varphi_0, 0.51\varphi_0, \varphi_0\}$ where the maximum value $\varphi_0 \equiv \pi(\frac{1}{2}-\nu)$. The magnitude of $\hat{h}$ has been scaled to make the large temperature behavior coincide: $h=\{0.34, 0.39, 0.58, 1\}$.

While the large temperature scaling (3) is the most universal feature of our results, we can also plot the full amplitude (1) as a function of temperature for given values of the parameters. We will introduce the various free parameters of the model below. Typical results are shown in Fig. 1. The most interesting observation is that for a given value of the critical exponent $\nu<\frac{1}{2}$ there is a range of possible behaviors at low temperature. While the curves can saturate, mimicking the usual Lifshitz-Kosevich behavior, it is also possible for the curve to reach zero temperature with a finite negative gradient or alternatively to exhibit a maximum before reaching zero temperature with a positive gradient. A maximum was reported experimentally in Ref. 22. In Ref. 22, it was further noted that an improved fit to the data could be achieved by modifying the Kosevich-Lifshitz formula.

The theories for which Eq. (1) will be shown to hold are described using the "holographic correspondence." We will not review the methodology in detail, introductions written for the condensed matter community can be found in Refs. 19 and 20, but rather summarize the physical properties of the theories in question.

The holographic correspondence allows a class of strongly interacting quantum field theories to be studied in a limit in which there are a large number of degrees of freedom per site. Unlike more traditional vector "large N" limits, the theories do not become weakly interacting in this limit, and might therefore be expected to capture aspects of interesting experimental systems that would otherwise elude theoretical control.

It was shown in Ref. 23, following earlier work in Refs. 24–26, that the fermion spectral densities in these theories exhibit a broad peak with a zero temperature dispersion relation at $k\sim k_F$ of the form

$$\frac{\omega}{v_F}+he^{i\theta}\omega^{2\nu}=k-k_F, \tag{5}$$

where $\{v_F,h,\theta,\nu,k_F\}$ are real constants. For $\nu<\frac{1}{2}$ the nonanalytic term $\omega^{2\nu}$ dominates at low frequencies, leading to non-Fermi liquid behavior. This non-Fermi liquid behavior is characterized by an emergent low energy, $\omega \ll \mu$, scale invariance with ν determining the dynamical critical exponent. For this reason we refer to our theories as quantum critical. It is a "metallic" quantum criticality in the sense that the momentum is scaled to k_F rather than zero. The case $\nu=\frac{1}{2}$ leads to the dispersion relation $\frac{\omega}{v_F}+he^{i\theta}\omega\log\omega=k-k_F$, which is precisely that of a marginal Fermi liquid.[27] For all $\nu\leq\frac{1}{2}$, the peak in the spectral density does not correspond to a stable quasiparticle excitation. This is because the width of the peak is always comparable to its height. Viewed as a pole in the spectral density in complex frequency space, its residue goes to zero as the pole hits the real axis at $k=k_F$.[23] In principle we could also study $\nu>\frac{1}{2}$, but here the linear term in Eq. (5) dominates at low energies and a more conventional behavior is expected. See however[23] for some curious properties of these cases.

Given that Eq. (5) does not describe a weakly interacting (stable) quasiparticle, one can anticipate that the contribution of the fermions to thermodynamic and transport quantities will not be simply that of a free fermion with dispersion Eq. (5). The correct way of computing in these systems was developed in Ref. 28, with the more mathematical aspects treated in Ref. 29. The essential step is to consider Eq. (5) as the singular locus of the fermion spectral density $\text{Im}\, G_R(\omega,k)$. It is easy to see that Eq. (5) has two types of singularities, a pole and then a branch cut emanating from $\omega=0$. While the pole describes the naïve "quasiparticle," both the pole and the branch cut will give contributions to, e.g., thermodynamic quantities.

This paper will be concerned with small but finite temperatures. At finite temperature, the branch cut of Eq. (5) is resolved into closely spaced poles. For $T,\omega\ll\mu$ one obtains[23] that the poles of $\text{Im}\, G_R(\omega,k)$ are given by solutions to

$$\mathcal{F}[\omega_\star(k)]=0, \tag{6}$$

where

$$\mathcal{F}(\omega)=\frac{k-k_F}{\Gamma\left(\frac{1}{2}+\nu-\frac{i\omega}{2\pi T}-i\alpha\right)}-\frac{he^{i\theta}e^{i\pi\nu}(2\pi T)^{2\nu}}{\Gamma\left(\frac{1}{2}-\nu-\frac{i\omega}{2\pi T}-i\alpha\right)}. \tag{7}$$

See e.g., Figure (3) of Ref. 28. The dimensionless constant α is related to the normalization of the current-current correlator.[23] While complicated, this formula is largely fixed by an emergent $SL(2,\mathbb{R})$ (or possibly even Virasoro) symmetry at energies $\omega\ll\mu$, suggesting perhaps validity beyond the specific holographic theories considered in Ref. 23. This emergent IR scaling symmetry is the quantum criticality referred to in the title of this paper. The only dimensionful scales in the theory are the chemical potential μ, magnetic field B, Fermi momentum k_F and temperature T. In Eq. (7) we have assumed that $\nu<\frac{1}{2}$ so that the linear in ω term in Eq. (5) can be dropped at low energies.

All of the poles given by Eq. (6) contribute to quantities of interest, even those that are a long way away from the real frequency axis. The key result of Refs. 28 and 29 was to express the contribution of the fermions to the free energy as a sum of contributions from these poles. The formula is

$$\Omega = \frac{eBAT}{2\pi c}\sum_{\ell}\sum_{\omega_\star(\ell)} \log\left(\frac{1}{2\pi}\left|\Gamma\left(\frac{i\omega_\star(\ell)}{2\pi T}+\frac{1}{2}\right)\right|^2\right). \quad (8)$$

Anticipating our interest in magnetic fields, we have given the free energy as a sum over Landau levels rather than momenta. The first term in Eq. (8) is the degeneracy of the Landau levels. The frequencies $\omega_\star(\ell)$ are obtained from $\omega_\star(k)$ in Eqs. (6) and (7) by the replacement $k^2 \to \frac{2\ell eB}{c}$. This replacement is precise in the limit $\frac{eB}{c} \ll k_F^2$ that we will be interested in. The formula (8) is not as exotic as it may appear; for instance, the free energy of a damped harmonic oscillator can be computed using essentially the same formula, with $\omega_\star$ again given by the poles of the retarded Green's function.[28,29] The appearance of $|\Gamma(ix+\frac{1}{2})|^2$ is a generalization of the Fermi-Dirac distribution to complex energies. If x is real then $|\Gamma(ix+\frac{1}{2})|^2=\pi \operatorname{sech} \pi x$, recovering the standard expression.

Our objective is to perform the sum (8) given Eq. (7) to obtain the magnetic susceptibility for general $T \sim \frac{eB}{m_\star c} \ll \mu$. The result for the leading oscillatory part of the susceptibility is stated in Eq. (1).

II. COMPUTATION

Our starting point is the formula for the fermionic contribution to the free energy, given in Eq. (8) in terms of the poles (6) of the fermion retarded Green's function. It will be useful to consider the dimensionless quantity

$$\hat{\Omega} \equiv \frac{2\pi c}{eBAT}\Omega = \mathrm{Re}\sum_{\ell}\sum_{x_\star(\ell)} 2\log\Gamma\left(x_\star(\ell)+\frac{1}{2}\right), \quad (9)$$

where we set

$$x = \frac{i\omega}{2\pi T}. \quad (10)$$

In the formula (7) defining the poles we will furthermore set

$$\hat{h} \equiv \frac{h e^{i\theta+i\pi\nu}(2\pi\mu)^{2\nu}}{\pi k_F} \equiv \bar{h}(\sin\varphi + i\cos\varphi), \quad (11)$$

so that $\{\hat{h},\bar{h},\varphi\}$ are now dimensionless. While in principle these parameters are determined by data in the UV by solving some ordinary differential equations numerically,[23] we will simply treat them as order one quantities, as we are more interested in parametrizing possible low energy physics. There is a restriction on φ that ensures that the poles are in the lower half frequency plane: $-\pi(\frac{1}{2}-\nu) < \varphi < \pi(\frac{1}{2}-\nu)$. Notice that the imaginary part of $\hat{h}$ is always positive.

Using all these expressions we can rewrite the sum over $\omega_\star(\ell)$ as a contour integral. Noticing that $\mathcal{F}(x)$ does not have poles, just zeroes in the right half plane [corresponding to the poles $\omega_\star$ of $G_R(\omega,k)$ in the lower half plane] we can write

$$\hat{\Omega} = \mathrm{Re}\frac{i}{\pi}\sum_{\ell}\int_{-1/4-i\infty}^{-1/4+i\infty} dx \log\Gamma\left(x+\frac{1}{2}\right)\frac{\mathcal{F}'(x)}{\mathcal{F}(x)}. \quad (12)$$

The contour was chosen such that it leaves the poles of $\frac{\mathcal{F}'(x)}{\mathcal{F}(x)}$ to the right and the branch cut of $\log\Gamma(x+\frac{1}{2})$ to the left. Implicitly we are also taking the contour to include a large semicircle in the right half plane. We will not need to evaluate the contribution from the semicircle explicitly, at a later step we will exchange the current sum over poles inside the contour for a sum of poles outside the contour (i.e., in the left-hand plane).

We would like to integrate (12) by parts, but this is complicated by the presence of the branch cuts from the logarithmic term. However, the derivative of $\hat{\Omega}$ with respect to the magnetic field can be safely integrated by parts to give

$$\hat{M} \equiv \frac{\partial\hat{\Omega}}{\partial B} = \mathrm{Re}\frac{1}{i\pi}\sum_{\ell}\int_{-1/4-i\infty}^{-1/4+i\infty} dx \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)}\frac{\partial_B\mathcal{F}(x,B)}{\mathcal{F}(x,B)}. \quad (13)$$

We will be interested in considering the periodic behavior in $\frac{1}{B}$ of this expression. Therefore, it is of use to Fourier transform the Landau level variable ℓ. We will perform a Poisson resummation to rewrite Eq. (13). The formula we use is

$$\sum_{\ell=0}^{\infty} f(\ell) = \sum_{k=-\infty}^{\infty}\int_{0^-}^{+\infty} dx f(x) e^{i2\pi kx}. \quad (14)$$

It is straightforward to apply this formula to Eq. (13), with the Landau levels going over $\ell=0,1,2,\ldots$ We obtain

$$\hat{M} = \mathrm{Re}\frac{1}{i\pi}\sum_{k=-\infty}^{\infty}\int_{-1/4-i\infty}^{-1/4+i\infty} dx \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} G(x,B,k), \quad (15)$$

where

$$G(x,B,k) \equiv \int_0^\infty d\ell \frac{\partial_B\mathcal{F}(x,B,\ell)}{\mathcal{F}(x,B,\ell)} e^{i2\pi k\ell} = \frac{ck_F^2}{2eB^2}\int_0^\infty \frac{du\, u^2 e^{i2\pi(ck_F^2/2eB)ku^2}}{u-\left[1+\pi\hat{h}\left(\frac{T}{\mu}\right)^{2\nu} S_\nu(x)\right]}. \quad (16)$$

in which we used the explicit form of Eq. (7), changed variables to $u=\sqrt{\frac{2eB\ell}{ck_F^2}}$ and set

$$S_\nu(x) = \frac{\Gamma\left(\frac{1}{2}+\nu-i\alpha-x\right)}{\Gamma\left(\frac{1}{2}-\nu-i\alpha-x\right)}. \quad (17)$$

Equations (15) and (16) appear to involve formidable sums and integrals. However, we can now neatly separate out the oscillating and nonoscillating parts of this expression. We will deform the contour in such a way that the integral follows a steepest descent path of the exponential term. The reason this helps is that the resulting integral is manifestly nonoscillating in $1/B$.

We therefore deform the integral in Eq. (16) by $u \to e^{ik/|k|\pi/4}u$. It is crucial to realize here that the contour needs to be rotated in opposite directions in the complex plane, depending on the sign of k, to guarantee convergence. The only possible obstructions to this contour rotation are either a contribution at infinity, which is absent in our case as the integrands decay exponentially if the paths are rotated in the correct direction, or if a pole is crossed as the contour is deformed. The expression (16) makes manifest that there is such a pole at $u=1+\pi\hat{h}(\frac{T}{\mu})^{2\nu}S_\nu(x)$. In the limit of physical interest, $T/\mu \to 0$, this pole is slightly off the real axis, for $0<\nu<\frac{1}{2}$, where our formulas are valid.

The exact position of the pole depends on the phase of $\hat{h}$ but it is always slightly above the real axis (this can easily be checked for the allowed range of values of $\hat{h}$ and $x \in -\frac{1}{4}+i\mathbb{R}$). The upshot is that for negative k we can rotate the contour and get

$$G(x,B,-|k|) = \frac{ck_F^2}{2eB^2}e^{-i(\pi/2)} \times \int_0^\infty du \frac{u^2 e^{-2\pi(ck_F^2/2eB^2)|k|u^2}}{u - e^{i(\pi/4)}\left[1+\pi\hat{h}\left(\frac{T}{\mu}\right)^{2\nu}S_\nu(x)\right]}. \tag{18}$$

This contribution is strictly nonoscillating[1] in $\frac{1}{|B|}$. Deforming the contour for positive k we pick up a contribution from the pole. Calculating the appropriate residue yields

$$\begin{aligned} G(x,B,|k|) &= G_{\text{non-osc.}}(x,B,|k|) + G_{\text{osc.}}(x,B,|k|) \\ &= G_{\text{non-osc.}}(x,B,|k|) + \frac{\pi i c k_F^2}{eB^2} \\ &\quad \times \left[1+\pi\hat{h}\left(\frac{T}{\mu}\right)^{2\nu}S_\nu(x)\right]^2 \\ &\quad \times e^{i2\pi(ck_F^2/2eB)|k|[1+\pi\hat{h}(T/\mu)^{2\nu}S_\nu(x)]^2}. \end{aligned} \tag{19}$$

The first term is nonoscillating and is the same as Eq. (18) with various factors of $e^{i\pi/4} \to e^{-i\pi/4}$. We are therefore left with the following oscillating contribution

$$G_{\text{osc.}}(x,B,k) = \Theta(k)G_{\text{osc.}}(x,B,|k|). \tag{20}$$

where $\Theta(k)=1$ for $k>0$ and $\Theta(k)=0$ for $k<0$. The $k=0$ term is also nonoscillating and does not concern us. We have thus performed the first of our integrals, insofar as obtaining the oscillating term is concerned.

The next integral to address is the x integral in Eq. (15). We will convert this integral into a sum over residues that are *outside* the original region of integration. That is, to the left of the imaginary axis. Doing this allows us to represent the integral as a sum of the residues of the poles of $\frac{\Gamma'(x+\frac{1}{2})}{\Gamma(x+\frac{1}{2})}$. These are located at $-\frac{1}{2}-n$ with $n=0,1,2,3,\ldots$ and have minus unit residue. Combining this operation with the result (20), our expression (15) becomes

$$\begin{aligned} \hat{M}_{\text{osc.}} &= \text{Re}\frac{2\pi c k_F^2}{ieB^2}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \\ &\quad \times \left[1+\pi\hat{h}\left(\frac{T}{\mu}\right)^{2\nu}S_\nu\left(-\frac{1}{2}-n\right)\right]^2 \\ &\quad \times e^{i2\pi(ck_F^2/2eB)k[1+\pi\hat{h}(T/\mu)^{2\nu}S_\nu(-1/2-n)]^2}. \end{aligned} \tag{21}$$

It is clear at this point that we have obtained sums over terms that both oscillate and decay in $1/B$. We can now take the physical $T/\mu \to 0$ limit keeping only leading terms determining the oscillations and exponential decay. The result is

$$\hat{M}_{\text{osc.}} = \frac{2\pi c k_F^2}{eB^2}\sum_{k=1}^{\infty}\sin\frac{\pi c k_F^2 k}{eB}\sum_{n=0}^{\infty}e^{-2\pi^2(ck_F^2/eB)(T/\mu)^{2\nu}k\,\text{Im}\,\hat{h}S_\nu(-1/2-n)}. \tag{22}$$

This last formula is essentially the result. To compute the magnetic susceptibility χ we have to reinsert the factors that relate Ω to $\hat{\Omega}$ in Eq. (9). Thus

$$\chi = -\frac{\partial^2\Omega}{\partial B^2} = -\frac{eAT}{\pi c}\hat{M} - \frac{eBAT}{2\pi c}\frac{\partial\hat{M}}{\partial B}. \tag{23}$$

For situations of physical interest we have $\frac{eB}{c} \ll k_F^2$ and therefore the leading result comes from the second term by acting with the derivative on the sine in Eq. (22). Focusing on the leading period, the $k=1$ term, this gives our main result, that we already quoted in Eq. (1), with

$$F_n(\nu) = 2\pi^2\,\text{Im}\,\hat{h}S_\nu\left(-\frac{1}{2}-n\right). \tag{24}$$

We also already noted in the introduction that the case $\nu=\frac{1}{2}$ is special. This is because the ratio of gamma functions in Eq. (17) simplifies in this case to give $F_n(\frac{1}{2})=2\pi^2\bar{h}(n+\frac{1}{2})$. The sum over n can then be done explicitly, to yield a result of the standard Lifshitz-Kosevich form (2).

In general, we cannot perform the sum over n in closed form. However, it is simple to check numerically that for all allowed values of the parameters, $F_n(\nu)$ is positive and monotonically increasing in n. Therefore at the high temperatures of primary interest we can keep only the first term in the sum in Eq. (22) or Eq. (1)given by $n=0$. This observation also implies that the $k=1$ term kept in Eq. (1) has an exponentially larger amplitude than the other terms in this regime. Thus we obtain, for general $\nu<\frac{1}{2}$, the non-Lifshitz-Kosevich scaling that we quoted in Eq. (3).

III. GENERAL FORMULA FOR QUANTUM OSCILLATIONS

We will now rederive the result (1) via a slick argument. The argument is quite general and we anticipate future applications. The method used is a generalization of that in

Refs. 2 and 29 and we will be brief in presentation.

The statement is that for any fermionic system satisfying assumptions to be given shortly

$$\Omega_{\text{osc.}} = \frac{eBAT}{\pi c}\text{Re}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{i2\pi k\ell_\star(n)}, \qquad (25)$$

where the $\ell_\star(n)$ are defined as the solutions to

$$\mathcal{F}[\omega_n,\ell_\star(n)] = 0. \qquad (26)$$

Here $\mathcal{F}(\omega,\ell)=0$ defines the singular locus of the fermion retarded Green's function in a magnetic field, $G_R(\omega,\ell)$. The fermionic Matsubara frequencies are $\omega_n=2\pi iT(n+\frac{1}{2})$. We assume for simplicity that there is a unique $\ell_\star(n)$, but it is simple to relax this assumption. It is clear that using Eq. (7) with $k^2=\frac{2\ell eB}{c}$, solving for $\ell_\star(n)$ as in Eq. (26) and plugging into Eq. (25) immediately reproduces our previous result (21).

The class of theories to which the formula (25) will most directly apply are those where the fermionic partition function can be expressed as the determinant of an operator $\mathcal{O}$ in a thermal Euclidean space. This certainly applies to free theories and to theories with classical holographic duals. In the latter case the determinant is in one extra curved spacetime dimension, but this does not make a difference to the argument. We assume that in a background magnetic field, the eigenvalues of the operator can be labeled by the quantum numbers ω_n and ℓ as well as any others. The type of reasoning in Ref. 29 is quickly seen to imply that we must have, up to UV contributions that can be dealt with systematically but which will not contribute to oscillations,

$$\Omega = -T\,\text{tr}\log\mathcal{O} = -\frac{eBAT}{\pi c}\text{Re}\sum_{\omega_n\geq 0}\sum_{\ell}\log[\ell-\ell_\star(n)]. \qquad (27)$$

The logic that leads to this expression is to separate the eigenvalues of $\mathcal{O}$ according to ω_n and ℓ. The contribution from positive and negative ω_n to the determinant are complex conjugates of each other[28,29] so we concentrate on the positive Matsubara frequencies. For a fixed ω_n we can deform the operator by letting $\ell\to\ell+\gamma$ and then match the zeros of the determinant of $\mathcal{O}_{n,\gamma}$ as a function of γ. Zeros arise whenever $\mathcal{O}_{n,\gamma}$ has a zero mode. This in turn occurs whenever the Euclidean Green's function has a pole at $\omega=\omega_n$, which we define to occur at $\ell+\gamma\equiv\ell_\star(n)$. The retarded Green's function is the analytic continuation of the Euclidean Green's function from the upper half frequency plane, thus connecting with our definition of $\mathcal{F}(\omega,\ell)$ appearing in Eq. (26). Writing $\det\mathcal{O}_{n,\gamma}\sim\Pi_\ell(\ell+\gamma-\ell_\star(n))$ and setting $\gamma=0$ gives Eq. (27).

Poisson resumming Eq. (27) using Eq. (14) and picking out the oscillatory part of the Fourier transform by rotating the contour in different directions for negative and positive k, in a similar way to how we did previously, then directly leads to Eq. (25). Only the rotation at positive k leads to a singularity contribution giving the oscillating term.

We now see that the formula (25) reproduces known expressions for free fermions. The nonrelativistic, spinless electron (the effect of spin is simply to multiply the answer by two in the limit $\frac{eB}{c}\ll k_F^2$) has

$$\mathcal{F}_{\text{non-rel.}}(\omega,\ell) = \frac{Be}{mc}\ell-\mu-\omega. \qquad (28)$$

It is trivial to solve for $\ell_\star(n)$ defined via Eq. (26). Plugging into Eq. (25), differentiating twice and performing the geometric series sum over n as previously leads to

$$\chi_{\text{osc.}} = \frac{2\pi AT\mu^2m^2c}{B^3e}\sum_{k=1}^{\infty}\frac{k\cos\left(2\pi k\frac{\mu mc}{Be}\right)}{\sinh\left(2\pi^2k\frac{Tmc}{Be}\right)}. \qquad (29)$$

This is literally the Lifshitz-Kosevich formula[2] in 2+1 dimensions, which we have derived rather painlessly. The fact that $\mathcal{F}$ is linear in ℓ in Eq. (28) makes the steps leading to Eq. (27) trivial in this case, there is no rewriting involved.

We can treat the spinless relativistic fermion similarly. In this case

$$\mathcal{F}_{\text{rel.}}(\omega,\ell) = m^2c^4+2Be\ell c-(\mu+\omega)^2. \qquad (30)$$

It is again immediate to solve for ℓ. Use of Eq. (25), the limit $T\ll\mu$, differentiation and summing a geometric series gives

$$\chi_{\text{osc.}} = \frac{\pi ATck_F^4}{2B^3e}\sum_{k=1}^{\infty}k\frac{\cos\left(\pi k\frac{ck_F^2}{Be}\right)}{\sinh\left(\pi^2k\frac{T\mu}{Bec}\right)}. \qquad (31)$$

We used the relation $k_F^2c^2=m^2c^4-\mu^4$. In the massless limit (or μ much larger than mc^2) this expression recovers our result (2) for the "marginal" non-Fermi liquid at $\nu=\frac{1}{2}$ if we choose $\bar{h}=1$.

The expression (25) is essentially the same as a general expression appearing in Ref. 14. In Ref. 14 the effects of interactions are incorporated into a renormalized self-energy for quasiparticles whose one loop contribution to the susceptibility is then computed. This is a controlled approximation if there are well-defined quasiparticles so that higher-order corrections that cannot be absorbed into the self energy are negligible. In the holographic theories studied here the self-energy due to strong interactions is captured by the propagation of the fermions on a nontrivial background spacetime, leading to the singular locus (7). Interactions between these fermions are suppressed by the "large N" limit in which the holographic computations are performed. Therefore, holography provides a controlled setting in which the self energy can be strongly renormalized to the extent that there are not well-defined quasiparticles and yet quantities such as the susceptibility can be computed with a determinant formula such as Eq. (25).

IV. MAGNITUDE OF OSCILLATIONS AND THE FERMI SURFACE

We need to check that our result could in principle be measured. For that purpose we compare the order of magnitude of the amplitude of the oscillating part to the nonoscillating part. We will pursue this calculation at low temperatures, where the oscillating signal is strongest. In this limit, we will see that the oscillating susceptibility strongly dominates over the nonoscillating part in the regime of interest $\frac{eB}{c} \ll k_F^2$ for $\frac{1}{6} < \nu < \frac{1}{2}$. This dominance is, of course, not a strict requirement for experimental detection. We first estimate the oscillating magnetization. At low temperatures all terms in the sum in Eq. (1) are important. In fact, the infinite tail of this sum dominates. Therefore, we can replace $S_\nu(-\frac{1}{2}-n)$ with its $n\to\infty$ limit, $S_\nu(-\frac{1}{2}-n)\to n^{2\nu}$. Because the quantity appearing in the sum is $\frac{T}{\mu}n$, we can replace the sum in n with an integral at leading order in $\frac{T}{\mu}$. Therefore the magnitude of Eq. (1) becomes

$$\frac{|\chi_{\rm osc}^{T\sim 0}|}{A} \sim \frac{ck_F^4 T}{eB^3}\int dn e^{-c\kappa_F^2/eB(T/\mu)^{2\nu}n^{2\nu}} \sim \frac{e^2\mu}{c^2k_F^2}\times\left(\frac{eB}{ck_F^2}\right)^{1/2\nu-3}. \tag{32}$$

It is interesting to rederive this last result from a different perspective that makes transparent the role of a Fermi surface. At low temperatures the susceptibility is most naturally written as a sum over ℓ, without Poisson resumming. We can start from the expression (13) for $\hat{M}$ and calculate χ by use of Eq. (23). As before, we can change the x integral to a sum over poles labeled by n. Once again, at zero temperature the tail of this sum dominates and we can substitute $\Sigma_n \to \int dn$ and $S_\nu(-\frac{1}{2}-n)\to n^{2\nu}$. The resulting integral can be performed analytically to leave a sum over ℓ that is similar to the expressions obtained in Ref. 28. This sum has a nonanalyticity at $\ell = \frac{ck_F^2}{2eB}$. Expanding the susceptibility at small $\frac{eB}{ck_F^2}$ using a generalized version of the Euler-Maclaurin formula,[31] the sum in ℓ becomes an integral plus contributions at the edges. The edge near the Fermi surface is responsible for the leading effect we are interested in. Explicitly

$$\begin{aligned}\frac{|\chi^{T\sim 0}|}{A} &\sim \sum_\ell \frac{\mu e}{cB} g\left(\frac{2eB\ell}{ck_F^2}\right)\left(1-\sqrt{\frac{2eB\ell}{ck_F^2}}\right)^{-2+1/2\nu} \\ &\sim \text{Analytic}(B) + \frac{e^2\mu}{c^2k_F^2}\left(\frac{eB}{ck_F^2}\right)^{1/2\nu-3},\end{aligned} \tag{33}$$

where $g(\cdot)$ is a dimensionless function that is regular at 1. The analytic terms give a generic expansion, with the constant term representing, for instance, Landau diamagnetism. This piece includes contributions that have not been captured by the poles in Eq. (7), as this formula has zoomed in on the low energy states near the Fermi surface. The second term is the leading contribution coming from the Fermi surface and agrees with the previous computation Eq. (32). From Eq. (33) we can see that the oscillating term strongly dominates the susceptibility for $\frac{1}{6} < \nu < \frac{1}{2}$.

Finally, we can check that the scaling Eq. (3) is potentially observable in an experimentally interesting regime without being exponentially suppressed by temperature. Setting all dimensionless parameters except for ν to be order unity, we can estimate the magnitude of the oscillations. Taking μ to be of order eV, T to be of order Kelvin and reinserting fundamental constants the exponent in our final result (1) is of order

$$\frac{ck_F^2}{\hbar eB}\left(\frac{k_BT}{\mu}\right)^{2\nu} \sim \frac{F_B}{B}\times(10^{-4})^{2\nu}, \tag{34}$$

where F_B is the frequency of the oscillations measured in Tesla. In measurements on the underdoped cuprates, for instance, $F_B/B \sim 10$,[3] and so the exponent is not too large for a wide range of values of ν.

V. DISCUSSION

Using the holographic correspondence we have obtained the amplitude of quantum oscillations in a family of strongly interacting quantum critical theories. Our expression (1) provides a theoretical template for possible violation of Lifshitz-Kosevich scaling of the amplitude with temperature due to strong interactions. We also found that at the marginal value of the critical exponent $\nu = \frac{1}{2}$, the Lifshitz-Kosevich result (2) survives the interactions. Our results are perhaps the most concrete yet to emerge from applications of holography to condensed matter physics. The scalings we have described could conceivably be found in systems of current experimental interest. The basic input into our computation was incoherent electronic excitations with dispersion (5), with $\nu \le \frac{1}{2}$, that may arise at strongly interacting metallic quantum criticality. Similarly, the most promising regions for observing a violation of Lifshitz-Kosevich scaling are near quantum phase transitions where the effective mass of the charged quasiparticles diverges. The onset of the divergence of quasiparticle mass is observed in quantum oscillations in both heavy fermion[13] and cuprate systems.[12] In the cuprates this should also occur as one crosses from the underdoped to the overdoped region. Unfortunately, a larger quasiparticle mass makes the oscillation signal smaller and harder to detect experimentally.

It will be important to generalize our computations to include disorder and to see to what extent the textbook Dingle scaling is modified. The dynamics of holographic theories with disorder has barely been studied.[30] Furthermore, while the singular loci (7) for the Green's function is the simplest following from the holographic correspondence,[23] it is likely not unique. As finite density dual geometries become available, it will be of interest to see to what extent our result (1) is modified.

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*hartnoll@physics.harvard.edu
†dhofman@physics.harvard.edu

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